Counting and Computing Rational Points on Surfaces

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Part I: The method of Chabauty and Coleman for curves

Rational points on curves of genus $g \geq 2$

Let $C/\mathbb{Q}$ be a smooth projective curve of genus $g \geq 2$ with Jacobian J.

- Mordell's conjecture 1922: $C(\mathbb{Q})$ is finite.
- Chabauty's theorem '41: If $\operatorname{rk} J(\mathbb{Q}) < g$, then $C(\mathbb{Q})$ is finite.
- Faltings's theorem '83: Mordell's conjecture is true.

Sketch of Chabauty's proof

Take $x_0 \in C(\mathbb{Q})$, if any. Embed C into J via $x \mapsto [x - x_0]$. Let $r = \operatorname{rk} J(\mathbb{Q})$.

- Let Γ be the p-adic closure of $J(\mathbb{Q})$ in $J(\mathbb{Q}_p)$. It is a p-adic Lie subgroup of $J(\mathbb{Q}_p)$.
- The theory of p-adic Lie groups implies that $\dim \Gamma \leq r$.
- Hence, $\dim \Gamma < g = \dim J$.
- $C(\mathbb{Q}_p)$ generates $J(\mathbb{Q}_p)$, so it is not contained in Γ.
- It follows that $C(\mathbb{Q}_p) \cap \Gamma$ is finite.
- Finally, note that $C(\mathbb{Q}) = C(\mathbb{Q}_p) \cap J(\mathbb{Q}) \subset C(\mathbb{Q}_p) \cap \Gamma$.

Chabauty-Coleman bound

Coleman reinterpreted $\Gamma \cap C(\mathbb{Q}_p)$ as zeros of p-adic analytic functions on $C(\mathbb{Q}_p)$ constructed by integrating differentials.

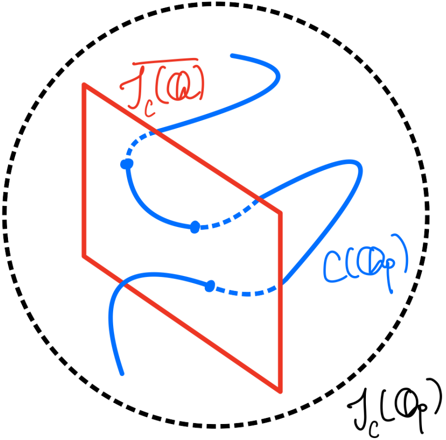
Theorem (Coleman 1985)

Let $C/\mathbb{Q}$ be a smooth projective curve of genus $g \geq 2$ and p a prime of good reduction for C. We assume:

- $p > 2g$,
- Dimension inequality: $\operatorname{rk} J(\mathbb{Q}) + 1 \leq g$,

then

$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_p) + 2g - 2.$$



Chabauty-Coleman beyond curves

What about a Chabauty-Coleman bound for X a higher dimensional variety?
We assume:

- X is hyperbolic and contained in an abelian variety A.
- Dimension inequality: $\operatorname{rk} A(\mathbb{Q}) + \dim X \leq \dim A$.

So far, only explored when $A = J$ the Jacobian of a curve $C \subset J$ and

$$X = \underbrace{C + C + \cdots + C}_{d \text{ times}} \qquad (\text{Essentially } \operatorname{Sym}^d C).$$

- Klassen '93: Finiteness on a p-adic open set.
- Siksek '09: Over number fields. Practical procedure for computations.
- Park '16: A conditional bound (not of Coleman type) under certain technical assumptions from tropical geometry.

Part II:
A Chabauty–Coleman bound for surfaces

Results over $\mathbb{Q}$ - also available over number fields

Notation: For a smooth variety V with good reduction at p, the reduction is V'.

Main Theorem(C-, Pasten '23)

Let X be a nice surface of general type in an abelian variety A of dimension 3 over $\mathbb{Q}$, and let p be a prime of good reduction for X and A.
We assume:

- $p > (\frac{128}{9}) \cdot (c_1^2(X))^2$,
- $\operatorname{rk} A(\mathbb{Q}) \leq 1$ (Dimension inequality),
- X' contains no elliptic curves over $\mathbb{F}_p^{alg}$ "finiteness".

Then

$$\begin{aligned}\#X(\mathbb{Q}) &\leq \#X'(\mathbb{F}_p) + \frac{p-1}{p-2} \cdot \left(p + 4p^{1/2} + 5\right) \cdot c_1^2(X) \\ &< \#X'(\mathbb{F}_p) + 4p \cdot c_1^2(X)\end{aligned}$$

Recall: $c_1^2(X)$: Self-intersection of the canonical divisor K_X of X.

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Then

$$\#X(\mathbb{Q}) < \#X'(\mathbb{F}_p) + 4p \cdot c_1^2(X)$$

Remark: Compare to Coleman's bound for curves:

$$\#C(\mathbb{Q}) \leq \#C'(\mathbb{F}_p) + c_1(C).$$

Remark

- Plenty of examples when $\mathrm{End}(A_{\mathbb{C}}) = \mathbb{Z}$, using a theorem of Chavdarov on primes of geometrically simple reduction. This works for instance on nice surfaces contained in the Jacobian of $y^2 = x^7 - x - 1$.
- Main Theorem is also available when $\dim A > 3$, but we need the existence of an ample divisor that satisfies a certain condition that depends on p, and an extra condition on A'.

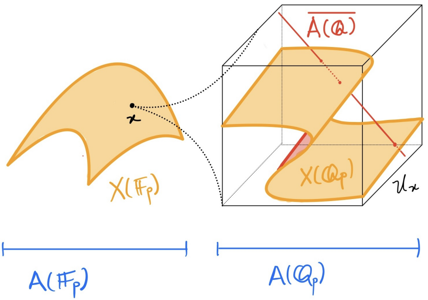
Idea of the proof

Setup

- $\Gamma = \overline{A(\mathbb{Q})}$ is a p-adic analytic 1-parameter subgroup of $A(\mathbb{Q}_p)$. Note:

$$X(\mathbb{Q}) = X(\mathbb{Q}_p) \cap A(\mathbb{Q}) \subset X(\mathbb{Q}_p) \cap \Gamma.$$

- **Reduction map:** red: $A(\mathbb{Q}_p) \to A'(\mathbb{F}_p)$. For each residue disk $U_x = \text{red}^{-1}(x)$ with $x \in X'(\mathbb{F}_p)$ we want to bound $\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x$.



How to bound $\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x$*?*

- Consider the analytic 1-parameter subgroup $\gamma : p\,\mathbb{Z}_p \to \Gamma \cap U_x$.
- Let f be a local equation for X on U_x. Then $f \circ \gamma(z)$ is a p-adic power series and

$$\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x \leq n_0(f \circ \gamma(z), 1/p).$$

- Write $f \circ \gamma(z) = \sum_n a_n z^n \in \mathbb{Q}_p[[z]]$.
- To bound $n_0(f \circ \gamma(z), 1/p)$ we have to find a small N such that $|a_N| \geq 1$.

This last requirement is **main difficulty** in the whole approach.

Large coefficient in low degree

Definition (ω-integral)

Let k be a field. Let Z, Y be k-schemes and $\omega \in H^0(Y, \Omega^1_Y)$. A k-morphism $\phi : Z \to Y$ is ω-integral if the composition

$$\phi^\bullet : H^0(Y, \Omega^1_Y) \to H^0(Y, \phi_*\phi^*\Omega^1_Y) = H^0(Z, \phi^*\Omega^1_Y) \to H^0(Z, \Omega^1_Z).$$

satisfies $\phi^\bullet(\omega) = 0$.

Notation: Let $k := \mathbb{F}_p^{alg}$ and $V_m := \operatorname{Spec} k[z]/(z^{m+1})$.
Suppose that there is $m < p$, such that $f \circ \gamma$ satisfies $|a_i| < 1$ for every $i \leq m$.
Then, there exists a closed immersion

$$\phi_m : V_m \to X_k,$$

at x which is w_i-integral for some $w_1, w_2 \in H^0(X_k, \Omega^1_{X_k/k})$ independent.

Large coefficient in low degree: the overdetermined method

- We need to bound the m such that the following holds:
  "There is a closed immersion

$$\phi : V_m \to X_k,$$

  at x which is w_i-integral for some $w_1, w_2 \in H^0(X_k, \Omega^1_{X_k/k})$."
- **The "overdetermined" bound:** We bound m in terms of the geometry of $D = \mathrm{div}(w_1 \wedge w_2)$.

Geometric bounds for ω-integral curves

Let S be a smooth surface over k, let $x \in S$, let $w_1, w_2 \in H^0(S, \Omega^1_{S/k})$ be independent over $\mathcal{O}_S$ and let $D := \operatorname{div}(w_1 \wedge w_2) = \sum_{j=1}^q a_j D_j$ with D_j irreducible curves and let $\nu_j : \widetilde{D_j} \to D_j \subseteq S$ be the normalizations.

Lemma (the bound on m)

Let $\phi : V_m \to S$ be a closed immersion supported at x. If ϕ is w_i-integral for both $i = 1, 2$ then

$$m \leq \sum_{j=1}^{q} \sum_{y \in \nu_j^{-1}(x)} a_j(\operatorname{ord}_y(\nu_j^{\bullet} w_i) + 1).$$

Bounds on U_x

A p-adic analysis argument implies that

$$\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x \leq n_0(f \circ \gamma(z), 1/p) \leq 1 + \frac{p-1}{p-2} \cdot m(x).$$

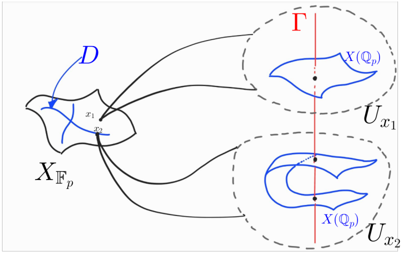
- Assume $x \notin \mathrm{supp}(D)$. We have $m(x) = 0$. Then

$$\#X(\mathbb{Q}_p) \cap \Gamma \cap U_x \leq 1 + \frac{p-1}{p-2} \cdot m(x) = 1.$$

- Assume $x \in \mathrm{supp}(D)$. **We use:** the Riemann hypothesis for singular curves, intersection theory computations, controlling singularities of D, etc.
  When we sum over all of these points, we get a bound in terms of the prime p and $c_1^2(X) = (D.D)$.

Last step

$$\#X(\mathbb{Q}_p)\cap\Gamma=\sum_{x\in X'(\mathbb{F}_p)}\#X(\mathbb{Q}_p)\cap\Gamma\cap U_x\leq\sum_{x\in X'(\mathbb{F}_p)}\left(1+\frac{p-1}{p-2}\cdot m(x)\right)$$
$$<\#X'(\mathbb{F}_p)+\frac{p-1}{p-2}\cdot\underbrace{(p+4p^{1/2}+5)\cdot c_1^2(X)}_{\text{terms } m(x) \text{ for } x\in D(\mathbb{F}_p)}.$$



Part III:
A refined Chabauty–Coleman bound for surfaces

Algebraic points on curves

Let C be a nice curve of genus $g \geq 3$. For $0 < d < g$, we define

$$C^{(d)} \qquad d\text{-th symmetric power of } C.$$

$C^{(d)}(\mathbb{Q})$ parametrizes degree s-points for $s \leq d$.
We have $\phi \colon C^{(d)} \to J_C$ and we define $W_d := \mathrm{Im}(\phi)$.

Example ($d = 2$)

When C is not hyperelliptic, ϕ is an isomorphism. Then

$$W_2(\mathbb{Q}) = \{\text{unorder pairs of rational points \& quadratic points}\}$$

Application: quadratic points

A consequence of Main Theorem is:

Corollary (C-, Pasten)

Let $C/\mathbb{Q}$ be a nice non-hyperelliptic curve of genus $g = 3$ whose Jacobian J satisfies $\operatorname{rk} J(\mathbb{Q}) \leq 1$. Let $p \geq 521$ be a prime of good reduction for C. Suppose that C' is not hyperelliptic and that $(C')^{(2)}$ does not contain elliptic curves over $\mathbb{F}_p^{alg}$. Then $C^{(2)}(\mathbb{Q}) = W_2(\mathbb{Q})$ is finite and

$$\begin{aligned}\#W_2(\mathbb{Q}) &\leq \#W_2'(\mathbb{F}_p) + 6 \cdot \frac{p-1}{p-2} \cdot \left(p + 4p^{1/2} + 5\right) \\ &< \#W_2'(\mathbb{F}_p) + 7.2 \cdot p.\end{aligned}$$

Hyperelliptic case

Example

If C is a nice hyperelliptic curve of genus $g \geq 3$, $C^{(2)}$ is the blow-up of W_2 at the origin. The exceptional divisor is

$$\nabla := \overline{\{[(x,y),(x,-y)] \colon (x,y) \in C\}}.$$

Then

$$W_2(\mathbb{Q}) = \{\text{unorder pairs of rat. pts. \& quadratic pts. lying outside } \nabla\}$$

Theorem (Balakrishnan, C- '25)

Let C be a hyperelliptic curve/$\mathbb{Q}$ of genus $g = 3$ whose Jacobian has rank ≤ 1. Let $p \geq 11$ be a prime of good reduction for C. Suppose that W_2' does not contain elliptic curves over $\mathbb{F}_p^{\mathrm{alg}}$. Then:

$$\#W_2(\mathbb{Q}) \leq \#W_2'(\mathbb{F}_p) + 2p + 12\sqrt{p} + 7.$$

We implemented an algorithm based on our method with H. Pasten.
Input: hyperelliptic curve C.
Output: An upper bound for the number of rational points of W_2.

Example

Let C be the curve defined by

$$C\colon y^2 = x(x^2+4)(x^2-4x-3)(x^2+4x+2).$$

We prove $W_2(\mathbb{Q}) = \{0_J\} \cup \{P_i : i = 1, \ldots, 6\}$, where

$$\begin{aligned}
P_1 &= (0,0) + 0_J,\\
P_2 &= (Q) + (Q^\sigma),\\
P_3 &= \left(2\sqrt{-1}, 0\right) + \left(-2\sqrt{-1}, 0\right),\\
P_4 &= \left(-2+2\sqrt{2}, 0\right) + \left(-2-2\sqrt{2}, 0\right),\\
P_5 &= \left(2+\sqrt{7}, 0\right) + \left(2-\sqrt{7}, 0\right),\\
P_6 &= \iota P_2,
\end{aligned}$$

$Q = \left(\frac{-13+2\sqrt{-14}}{9}, \frac{12560-7045\sqrt{-14}}{2187}\right)$ and Q^σ is the Galois conjugate of Q.

Notice that for $p = 5$ and $z = \overline{(0,0)} + \overline{(1,0)} \in W_2(\mathbb{F}_5)$, we have that

$$\{P_2, P_6, (2\sqrt{-1}, 0) + (0,0)\} \subset W_2(\mathbb{Q}_5) \cap \overline{J(\mathbb{Q})} \cap U_z.$$

The annihilator of $J(\mathbb{Q})$ under the integration pairing is spanned by

$$\omega_1 = (4 + 5 + 4 \cdot 5^3 + O(5^4))\frac{dx}{y} + (3 + 2 \cdot 5 + O(5^4))\frac{xdx}{y},$$
$$\omega_2 = (4 + 4 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + O(5^4))\frac{dx}{y} + (3 + 2 \cdot 5 + O(5^4))\frac{x^2dx}{y}.$$

So, we have

$$(w_1 \wedge w_2) = (4x_1x_2 + 2(x_1 + x_2) + 3)\frac{d(x_1x_2) \wedge d(x_1 + x_2)}{y_1y_2}.$$

Then $D = Z_{W_2}(4x_1x_2 + 2(x_1 + x_2) + 3)$. Notice that $z \in D$.

We prove that:

- z is a singular point of D.
- z has two preimages z_1 and z_2 in the normalization map ν.
- For $i = 1, 2$ we have
$$\mathrm{ord}_{z_i}(\nu^\bullet w_1) = 0$$

Then
$$m(z) \leq \sum_{j=1}^{q} \sum_{y \in \nu^{-1}(x)} a_j(\mathrm{ord}_y(\nu^\bullet w_1) + 1) = 2.$$

Finally, we have
$$\# W_2(\mathbb{Q}_5) \cap \overline{J(\mathbb{Q})} \cap U_z \leq 1 + \frac{4}{3} m(z) = 3.\overline{6}$$

Thank you very much for your attention!